Incompatibility and Inference as Bases of Logic

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Abstract: That logic can be based merely either on the concept of inference, or on that of incompatibility has been already shown. The question is whether such austere foundations predetermine the kind of logic we reach in such a way. In this paper we show that in the case of logic based on inference the natural outcome is intuitionist logic, while we can reach also classical logic (if we sacrifice naturalness). However, in case of logic based on incompatibility the outcome is not really optional: the resulting logic is classical and there is no obvious way how to reach intuitionist logic.

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It is not too controversial to say that we can base logic solely on the concept of inference. To indicate how this can be done, let us introduce an *inference structure* as an ordered pair, $\langle S, \vdash \rangle$, where where S is a set (of "sentences" or "formulas") and $\vdash \in \mathcal{P}(S) \times S$ is a relation between subsets of S and elements of S ("the relation of inferability"), such that[2]

($\vdash$1) $X, A \vdash A$,

($\vdash$2) if $X, A \vdash B$ and $Y \vdash A$, then $X, Y \vdash B$.

Then we can define incompatibility—let us call it *pseudoincompatibility* (for it is defined on the basis on inference and opinions on how this definition is successful may differ) and denote it be the sign $\triangle$—in the following way:

$$\triangle X \equiv_{\text{Def.}} X \vdash A \text{ for every } A.$$

Then we can define conjunctions, disjunctions etc. in the way pioneered by Koslow (1992): an element B of S is called *a negation* of an element A of S iff the following two conditions hold

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[2]Of course we could also consider "substructural" versions of inference structures based on rejecting some of these conditions. However, we do not do this in this paper.

(1) $\triangle A, B$,

(2) if $\triangle A, D$, then $D \vdash B$.

The first condition states that the negation of A is pseudoincompatible with A, whereas the second one states that it is the *minimal* element of S with this property: B is inferable from any other element of S which is also incompatible with A.[3] It follows that any two negations of A are equivalent in the sense that they are interinferable. Note also that in general the negation of a given element of S need not exist.

The situation is, of course, different when we take S to be generated from a basic vocabulary by means of some grammatical rule and if we introduce a specific negation sign producing, for every element A of S, its negation $\neg A$, like in common languages of logic. In such cases (1) and (2) directly stipulate the behavior of this new element of S:

($\neg$1) $\triangle A, \neg A$,

($\neg$2) if $\triangle A, D$, then $D \vdash \neg A$.

All other logical operators can be introduced in a similar vein

($\wedge$1) $A \wedge B \vdash A$ and $A \wedge B \vdash B$,

($\wedge$2) if $D \vdash A$ and $D \vdash B$, then $D \vdash A \wedge B$,

($\vee$1) if $A \vdash D$ and $B \vdash D$, then $A \vee B \vdash D$,

($\vee$2) if (if $A \vdash D$ and $B \vdash D$, then $E \vdash D$), then $E \vdash A \vee B$,

($\rightarrow$1) $A, A \rightarrow B \vdash B$,

($\rightarrow$2) if $A, D \vdash B$, then $D \vdash A \rightarrow B$.

Thus, Koslowian definitions give us every logically complex sentence as a minimum of a certain propositional function.

It might be interesting to consider a slight modification of Koslow's definition of disjunction:

[3] This presupposes that we read "$D \vdash B$", in effect, as "$B \leq D$".

($\vee 1'$) $A \vdash A \vee B$ and $B \vdash A \vee B$,

($\vee 2'$) if $A \vdash D$ and $B \vdash D$, then $A \vee B \vdash D$.

This disturbs the uniformity of Koslow's logic in that disjunction is no longer defined as the *minimum* of a propositional function (but rather a *maximum* of one), but it gives our definition of logical operators a more explicitly algebraic flavor: conjunction can be seen as *supremum* and disjunction as *infimum*. Anyway, the logic we reach in this way (both in the Koslowian, and in the modified one), not surprisingly, is intuitionistic.

Is there a way of reaching also classical logic in terms of inference? Yes, there is; it is enough to modify the definition of negation in the following way:

($\neg 1$) $\triangle A, \neg A$,

($\neg 2'$) if $\triangle \neg A, D$, then $D \vdash A$.

This is a definition not so neat as the previous one, but it does yield us classical logic. This is easily seen, for now we have

if $\triangle \neg A, \neg\neg A$, then $\neg\neg A \vdash A$

as an instance of ($\neg 2'$), and as the antecedent is an instance of ($\neg 1$), we have the consequent

$\neg\neg A \vdash A$

which is nothing else than the intuitionistically notoriously invalid law of double negation elimination.

So the *natural* outcome of basing logic on inference is intuitionistic logic; but if we are willing to sacrifice naturalness, we can reach classical logic as well.

Now turn your attention to logic based on incompatibility. The framework in which we can study this kind of logic is that of an *incompatibility structure*, which is an ordered pair $\langle S, \perp \rangle$, where S is a set and $\perp \in \mathcal{P}(S)$ is a set of subsets of S, such that

($\perp$) if $\perp X$ and $X \subseteq Y$, then $\perp Y$.

Here we can define inference—call it *pseudoinference* and denote it by $\triangleright$—as follows:

$X \triangleright A \equiv_{\text{Def.}} \perp Y, A$ implies $\perp Y, X$ for every Y.

How it is possible to define logical operators in this setting was shown by Brandom (2008):

$(\neg)$ $\perp\neg A, X$ iff $X \triangleright A$,

$(\wedge)$ $\perp X, A \wedge B$ iff $\perp X, A, B$.

It is easy to see that this definition gives us classical logic: as certainly $A \triangleright A$, $(\neg)$ gives us

$\perp\neg A, A$,

and as it also gives us

if $\perp\neg A, \neg\neg A$, then $\neg\neg A \triangleright A$,

we have

$\neg\neg A \triangleright A$.

Now the question is: is the situation similar to the previous one, in that though the most *natural* outcome of basing logic on incompatibility is classical logic, it would be possible to reach also intuitionistic one?

In light of the previous considerations concerning logic based on inference, it may seem that we might be able to reach intuitionistic logic by replacing $(\neg)$ by

$(\neg')$ $\perp A, X$ iff $X \triangleright \neg A$.

This is what I conjectured in (Peregrin, 2011). Since then, however, I delved deeper into the problem and I have found out that this impression is wrong (see Peregrin, 2015). Even if we replace $(\neg)$ by $(\neg')$, we still have classical logic. For suppose that

$\perp A, X$.

Then, according to $(\neg')$,

$X \triangleright \neg A$,

and hence, unpacking the definition of $\triangleright$,

$$\text{if } \bot\neg\neg A, \neg A, \text{ then } \bot\neg\neg A, X.$$

And as it is the case that

$$\bot\neg\neg A, \neg A$$

(for according to $(\neg')$, this is equivalent to $\neg A \triangleright \neg A$),

$$\bot\neg\neg A, X.$$

Hence the assumption $\bot A, X$ yields us $\bot\neg\neg A, X$, which is nothing else than $\neg\neg A \triangleright A$.

Why is this? What plays the crucial role is the definition of (pseudo)inference in terms of incompatibility.

First issue to realize is that (as I was reminded by Peter Milne), the incompatibilities of intuitionistic logic are *the same* as those of classical logic. This follows from the Glivenko Theorem which states that A is a theorem of classical logic iff $\neg\neg A$ is a theorem of intuitionistic one. It follows that $\neg A$ is a theorem of classical logic iff it is a theorem of intuitionistic one: for $\neg A$ is a theorem of classical logic iff $\neg\neg\neg A$ is a theorem of intuitionist logic, which in turn holds iff $\neg A$ is a theorem of intuitionist logic (as $\neg\neg\neg A \leftrightarrow \neg A$ is a theorem of intuitionistic logic). Further it follows that if $X \vdash A \wedge \neg A$, hence if $\triangle X$, in classical logic, then the same holds in intuitionist logic. The reason is that if $X \vdash A \wedge \neg A$ in classical logic, then, as the logic is compact, $X^* \vdash A \wedge \neg A$ for some finite subset X^* of X, then $\wedge(X^*) \vdash A \wedge \neg A$, where $\wedge(X^*)$ is the conjunction of the elements of X^*, then $\vdash \neg \wedge (X^*)$, and hence $\vdash \neg \wedge (X^*)$ also in intuitionist logic and thus $X \vdash A \wedge \neg A$ is also in intuitionist logic.

Hence starting from incompatibility we should not expect that we will be able to differentiate classical and intuitionist logic. But this does not yet explain why we get classical, rather than intuinionistic logic. Assume we base logic on inference: then we can define pseudoincompatibility, $\triangle$, and then pseudinference, $\blacktriangleright$, based on this pseudoincompatibility:

$$X \blacktriangleright A \equiv_{\text{Def.}} \triangle Y, A \text{ implies } \triangle Y, X \text{ for every } Y,$$

It is not necessary that $\vdash$ and $\blacktriangleright$ coincide; and indeed in intuitionistic logic they do not. However, if you start from incompatibilities, then the *only* way to define inference (at least the only minimally reasonable way I can think of) is to make $\triangleright$ directly into $\vdash$, which is the way of classical logic. There is no way to deviate $\vdash$ from $\triangleright$, as intuitionistic logic requires.

Why is it the case that in classical logic, but not in intuitionistic logic, $X \blacktriangleright A$ brings about $X \vdash A$? Suppose $X \blacktriangleright A$. This is to say that for all Y, if $\triangle Y, A$, then $\triangle Y, X$. As $\triangle Y, A$ iff $Y \vdash \neg A$ (both in intuitionistic and classical logic), it follows that if $Y \vdash \neg A$, then $\triangle Y, X$. Since $\neg A \vdash \neg A$, it further follows that $\triangle \neg A, X$. And in classical logic, though not in intuitionistic one, it follows that $X \vdash A$.

To sum up, we can conclude that while it is arguably possible to base logic both on the sole concept of inference, and on the sole concept of incompatibility, the nature of the ensuing logics is different. In the first case, the "natural way" leads us directly to intuitionist logic, but there is an optional alternative, the "not-so-natural" way which results into classical logic. On the other hand, if we take incompatibility as our base concept, we are on the way to classical logic. The reason is that there does not seem to be a way which would build inference out of incompatibility that would not already instill it with the essence of classical logic.

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