Many-valued logic or many-valued semantics?

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The separation problem

There have been, I am afraid, almost as many answers to the question *what is logic?* as there have been logicians. However, if logic is not to be an obscure "science of everything", we must assume that the majority of the various answers share a common core which does offer a reasonable delimitation of the subject matter of logic.

To probe this core, let us start from the answer given by Gottlob Frege (1918/9), the person probably most responsible for modern logic: the subject matter of logic is "truth", and especially its "laws"[1]. How should we understand the concept of "laws of truth"? The underlying point clearly is that the truth/falsity of our statements is partly a contingent and partly a necessary, lawful matter: that "Paris is in France" is true is a contingent matter, whereas that "Paris is in France or it is not in France" is true is a necessary matter (let us, for the time being, leave aside the Quinean scruples regarding the delimitation of necessarily true statements). Logic,then, should focus on the statements that are true as a matter of law (i.e. necessarily), or, more generally, the truth of which "lawfully depends" on some other statements (i.e. which are true as a matter of law provided these other statements are true).

This renders Fregean laws of truth as, in general, a matter of "lawful truth-dependence" - i.e. of entailment or inference (again, let us now disregard any possible difference between these two concepts). This yields a conception of logic as a theory of entailment or inference, a conception which looms behind many other specifications of the subject matter of logic and which, I think, is ultimately correct.

However, we can also see the logician – and this is the view we will stick to here – as trying to separate true sentences from false ones; or, equivalently, to map sentences onto *truth* and *falsity*. Let us first consider the case of a non-empirical language with a single, definite truth valuation – like the language of Peano arithmetic. Here, as each of the sentences has a *fixed* truth value, the term "separation" can be taken quite literally. If we take a language of a logic, i.e. a language which is not fully interpreted (only logical constants have fixed

[1] Frege (1918/9, p. 58): "Wie das Wort 'schön' der Ästhetik und 'gut' der Ethik, so weist 'wahr' der Logik die Richtung. Zwar haben alle Wissenschaften Wahrheit als Ziel; aber die Logik beschäftigt sich noch in ganz anderer Weise mit ihr. Sie verhält sich zur Wahrheit etwa so wie die Physik zur Schwere oder zur Wärme. Wahrheiten zu entdecken, ist Aufgabe aller Wissenschaften: der Logik kommt es zu, die Gesetze des Wahrseins zu erkennen."

meanings, all other expressions are parametrized) or a fully-interpreted, but empirical language, the task of the logician will no longer be literally separation, or specification of a single truth-valuation, but rather delimitation of a range of all those truth-valuations which are acceptable. (Here it is where the fact that logic is after "laws" comes to the open: for we are *not* interested in the differences between valuations which do not affect *logical* truth, especially those which are a contingent matter.)

Hence the task of the logician, viewed from this perspective, is the delimitation of the range of acceptable truth-valuations of the sentences of the given language – taking note of all the "lawful" features of the separation of true sentences from false ones. Let us call this the *separation problem*. Now probably the most crucial fissure within modern logic can be pictured as arising from disagreements over how to approach this separation, and which tools are available to the logician engaging within it. On the one hand, some logicians urge that the only meaningful way of dealing with a separation of this kind is *effecting* it – presenting rules the recursive application of which allows actual separation of truth and falsity. Here the only tools available to the logician are *explicit rules*. (Putting the concept of "rule" or of "effecting" to closer scrutiny will then expose the altercations separating various subcamps of the "effective" camp – such as those of the intuitionists, finitists etc.)

On the other hand, other logicians have argued that because in some cases we *can* consider a separation without being able to effect it (*viz.* the separation of those sentences of PA that are true [in the standard model] from the false ones), we should sldo deal with the separation problem in various 'nonexplicit' ways. (From this viewpoint, model theory, as contrasted with proof theory, is precisely a way of treating of [the existence of] various separations without being able to effect them.)

In this paper I will not address this dispute, but simply take for granted that the logicians of the latter camp do have a point. (Perhaps it is an irrelevant point for what the former camp takes as the crucial task of logic, but nevertheless it is a point on a wider conception of logic.) Hence, considering the language of classical propositional calculus (and consequently the part of natural language which it purports to regiment), we may say that its "laws of truth" can be formulated quite transparently, as follows:

(i) $\neg A$ is true iff A is not true
(ii) $A\wedge B$ is true iff A is true and B is true
(iii) $A\vee B$ is true iff A is true or B is true
(iv) $A\rightarrow B$ is true iff A is not true or B is true

Every truth-valuation which fulfills these constraints is acceptable and every acceptable truth-valuation does fulfill them. But, of course, complications arise as soon as we abandon the calm waters of the classical propositional calculus.

The compositionalization problem

It is clear that sentences of a logical language are infinite in number only potentially: what is directly given is a finite vocabulary and a finite set of recursively applicable grammatical rules. Likewise, also any assignment of values to sentences cannot but exist as generated by an assignment of some values to the words and by a stipulation of how the values of outputs of the rules are determined by the values of their inputs. In other words, independently of how "effective" our conception of logic is to be, we should require that any valuation of the sentences worth taking into account be *compositional*. Hence we can say that w.r.t. a language such as that of PA, the task of logic is to provide a *compositional* specification of its truth-valuation.

In view of this, the situation becomes much more complicated already when we move from the propositional to the (classical) predicate calculus (or, for that matter, to the language of PA). The trouble is that the acceptable truth-valuations of this language cannot be delimited in terms of constraints of the kind of (i)-(iv) above because the acceptable valuations are *not* necessarily compositional. As a quantified sentence need not contain any subsentence (but only a sub*formula*), there is nothing on which its truth-value can be rendered as depending. Tarski's well-known, ingenious solution to this consists in descending from *truth* to *satisfaction* – making a compositional theory of satisfaction and deriving a theory of truth from it. The moral seems to be that where there is no compositional theory of truth, we can make a compositional theory of something else, provided this something else is then capable of yielding us truth.

Returning to propositional calculus, we can see that there the possibility of giving a compositional account of truth vanishes once we cross the boundaries of classical logic. Consider modal logic: it is clear that the truth-value of $\Box A$ is not uniquely determined by that of A. The well-known solution here is to make a compositional theory of intensions (the class of possible worlds denoted by $\Box A$ is uniquely determined by that denoted by A) and to let the theory of truth be parasitic upon it (A is true within the actual world iff the world is an element of the set denoted by A[2]).

Similarly, let us consider the following axiomatic system (which is known to axiomatize Łukasiewicz's three-valued propositional calculus):

$A\rightarrow(B\rightarrow A)$

$(A\rightarrow B)\rightarrow((B\rightarrow C)\rightarrow(A\rightarrow C))$

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[2] Of course we need not be able to identify the actual world among all the other logically possible ones – being able to do this would mean to be omniscient. However, as logic is not interested in the contingent aspect of truth, this is not a task a logician would face.

$(\neg A\rightarrow\neg B)\rightarrow(B\rightarrow A)$

$\underline{((A\rightarrow\neg A)\rightarrow A)\rightarrow A}$

$A, A\rightarrow B \,/\, B$

Can we semantically separate the sentences marked as true by this system (i.e. theorems) from the false ones?

It is again clear that the truth-valuations appropriate for this logic are not compositional. Does this mean that the answer to the question about semantic separability is negative? Clearly no; for we know that we can define a compositional valuation of sentences, with its range consisting of *three* values, which assigns one of the values to all and only true sentences. Similarly, for some other logics we may need four values and for still other (modal, fuzzy) we may need an infinite number of values[3].

The moral seems to be that *the compositional account for* (*the laws of*) *truth may necessitate more values than just* truth *and* falsity. The point, as we have just seen, is that insofar as logic is committed to separating truth from falsity and to doing it in a compositional manner, its helpful method can consist in finding mappings of statements on some other values which: (1) are compositional; and (2) are readily transformable into truth-valuations. Hence the separation problem yields what can be called the *compositionalization problem*: the problem of finding a compositional mapping 'underlying' a given truth-valuation. The question then is what is the status of the entities constituting the range of the underlying mapping.

Let us consider in detail how the compositionalization problem is solved in the case of the simplest modal logic S5; i.e. logic with the axioms

$A\rightarrow(B\rightarrow A)$
$(A\rightarrow(B\rightarrow C))\rightarrow((A\rightarrow B)\rightarrow(A\rightarrow C))$
$(\neg A\rightarrow B)\rightarrow((\neg A\rightarrow\neg B)\rightarrow A))$
$\Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B)$
$\Box A\rightarrow A$
$\underline{\Diamond A\rightarrow\Box\Diamond A}$
$A, A\rightarrow B \,/\, B$
$A \,/\, \Box A$

The basic trouble is, we saw, that the truth value of $\Box A$ is in general *not* uniquely determined by that of A: if A is false, then $\Box A$ is bound to be false, too; but if A is true, then $\Box A$ may be in some cases also true, whereas in other cases false. From this viewpoint, it would seem that it would be enough to split the truth value *truth* into the values *necessary truth* and *contingent*

[3] This is not, of course, to say that we always *first* have a separation delimited by axiomatic means and *then* we seek its rendering by semantic ones. Many logics were first devised in purely semantic terms and some do not allow for an axiomatic treatment at all.

truth. Then we can say that $\Box A$ is necessarily true iff A is necessarily true and it is false otherwise:

A	$\Box A$
NT	NT
CT	F
F	F

However, having done this refinement, we would have to rewrite the truth tables for the other operators for the new values. In particular, we would have to say whether it is *contingent truth* or *necessary truth* on which negation maps falsity. And it seems that the only reasonable answer leads via the splitting of *falsity*. Once this value is split up into *contingent falsity* and *necessary falsity*, we can say that the negation of *contingent falsity* is *contingent truth* and that of *necessary falsity* is *necessary truth*:

A	$\neg A$
NT	NF
CT	CF
CF	CT
NF	VT

The table for necessity then can easily be amended accordingly:

A	$\Box A$
NT	NT
CT	NF
CF	NF
NF	NF

But this is far from the end – for now we must rewrite the tables for conjunction, disjunction etc. Take disjunction – how will the table for it look like with the new values? It is clear that the disjunction of *necessary truth* with anything will yield *necessary truth*; and that the disjunction of *necessary falsity* with anything will yield the anything. It is also clear that the disjunction of a truth with anything will be a truth and that the disjunction of two falsities will be a falsity. What remains unclear is, however, the precise value of the disjunction of two contingent values:

$A \vee B$		B			
		NT	**CT**	**CF**	**NF**
A	**NT**	NT	NT	NT	NT
	CT	NT	?T	?T	CT
	CF	NT	?T	?F	CF
	NF	NT	CT	CF	NF

Take two *contingent truths* – it is clear that their disjunction will be a *truth* – but *necessary*, or *contingent*? Intuitively, it would seem that this value is again *not* determined uniquely: the disjunctions of some kinds of contingent truths will be true only contingently, whereas some other kinds (e.g. the disjunctions of a contingent truth with its negation) will be true necessarily.

Hence what we obviously need, if we want to reach a compositional valuation, is a further refinement – we must split up contingent truth (and similarly contingent falsity) into still more values. But this time the situation is rather tricky: we cannot say that we simply split *contingent truth* into two values, one such that its disjunction with *contingent truth* will yield *contingent truth*, whereas the other will yield *necessary truth*. The idea rather is that the values of some pairs of sentences are 'complementary' (their disjunction yields necessary truth), whereas those of others are not (their disjunction yields contingent truth). As a result, we gain a more or less nontrivial Boolean algebra of values, which can be represented as the well-known algebra of sets of 'possible worlds' (possibly with the superstructure of an 'accessibility relation')[4].

Technical aspects of compositionalization

Before we turn to the philosophical question, let us briefly consider the technical side of the compositionalization problem. It can be generally formulated in the following way: given a part-whole system *P* such that all its elements are parts of the members of some distinguished set *S* of its wholes (the system can be seen as a certain finitely generated many-sorted algebra, cofinal in a distinguished sort of its), and a mapping *f* of *S* on the set {**T**, **F**}, we have to find a mapping *g* of the carrier of *P* on a set *R* such that

(i) *g* is a homomorphism (this is the requirement of compositionality – for it stipulates that for every operation *O* of the algebra there is a function *O** such that for every $x_1, ..., x_n$ from the domain of *O* it is the case that $g(O(x_1, ..., x_n)) = O^*(g(x_1), ..., g(x_n))$; and

(ii) there is a partial mapping *h* of *R* on {0,1} such that $h(g(x)) = f(x)$ for every *x* from *S*.

(In the simplest case, which corresponds to the case of propositional logic considered here, *S* may coincide with the whole carrier of *P*).

Can we compositionalize any given mapping in this way? A moment's reflection reveals that the identity mapping *i*, mapping every statement on itself, is a compositionalization of any given mapping *f* (indeed, *i* is trivially a homomorphism for *O** being simply *O*; and *h* is *f* itself). Hence this problem, as it stands, is solved trivially. But we might consider adding a further natural requirement: we might require that *R*, the set of values of the compositional

[4] See Peregrin (in print).

mapping underlying the one to be compositionalized, be *the smallest possible*. And it turns out that though the problem is still generally solvable even in this strengthened form, it is no longer utterly trivial.

Note, first, that the problem, as just formulated, is 'structural' - the answer cannot but be of the "up to isomorphism" kind. We minimalize the cardinality of a range; we do not care about its nature. Given this, we can replace every mapping f which we might want to consider by the mapping f^* which maps every sentence s on the class $\{s' \mid f(s') = f(s)\}$. But any such function is uniquely determined by its kernel, i.e. the relation $K_{f^*} = \{\langle x,y \rangle \mid f^*(x) = f^*(y)\}$. This means that we can recast the problem of compositionalization of functions as a problem of compositionalization of equivalences – in effect as the problem of finding a 'minimal' congruence containing a given relation. And as congruences are closed to intersection, this problem is easily solvable: the compositionalization $C(E)$ of a given equivalence relation E is the intersection of all congruences containing it; and the compositionalization of a given function f is hence – up to isomorphism – the natural mapping of S on its quotient according to $C(E)$, i.e. the function mapping s on $\{s' \mid s'\ C(K_f)\ s\}$[5].

This mathematically rather trivial problem is closely related to some less trivial ones. First, there is the *extension* problem addressed recently by Hodges (2001) and Westertåhl (2004). This is the problem of extending a partial 'meaning assignment' to the whole language – seeking a total mapping which would be compositional and which would be an 'extension' of the given partial mapping[6]. From the philosophical viewpoint, however, it is more interesting to think not about the constitution of further meanings out of some already given meanings, but rather of constitution of meanings out of non-meanings, in particular out of truth (more about this later).

Another related problem is that of *algebraization* – finding an algebraic semantics for an axiomatically given logic (see, e.g., Font et al., 2001). As Tarski, Henkin and others showed, algebras underlying such semantics can usually be built from the language of the logic in question: we can simply take the carriers of the algebras as consisting of the equivalence classes of expressions according to some suitable equivalence, especially a relation of intersubstitutivity *salva* logical equivalence. In effect, we are aiming precisely at the quotient according to the largest congruence w.r.t. truth or truth-dependence.

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[5] I discussed this in greater detail in Chapter 4 of Peregrin (2001).

[6] It need not be an *extension* in the sense of *superset* – furnishing new expressions with meanings may necessitate also a *refinement* of the original meaning-assignment.

'Compositional decomposition'

What is the nature of the new values necessitated by the compositionalization of truth-valuations? Are they new truth-values?

In a recent paper, Caleiro et al. (2005) argue vigorously for the thesis, tabled long ago by Suszko (1977), that there are only *two* truth values, backing it by the thesis that any many-valued semantics can be reduced to a two-valued one. The simplest version of what the authors call "Suszko Reduction" is rather trivial: it says that given a valuation *V* of sentences and a division of the values of the range of this valuation into 'distinguished' and 'non-distinguished', i.e. in effect a function *D* mapping the range of *V* on {**T**, **F**}, then there is a truth-valuation *T* mapping a sentence on **T** iff the sentence is mapped on a distinguished value by *V*. (Indeed, *T* is simply $D \circ V$, the composition of *D* and *V*.) We will call the values which constitute the range of *V* and the domain of *D* *intermediary values*.

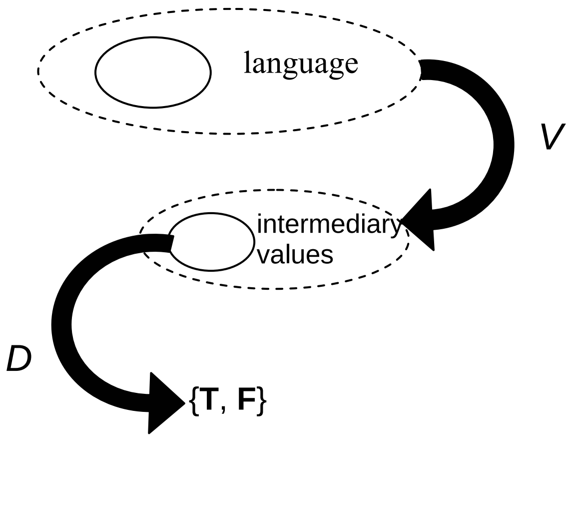
And from what has been said above, it should also be clear why this result is not very interesting: we have good reasons to restrict our interest to *compositional* (truth-)valuations and there is no guarantee that *T* will be compositional even if *V* is. Therefore it seems that the troublemaker must be the other mapping: if *V* is 'well-behaved', but $D \circ V$ is not, then it would seem that the 'non-well-behaved' one must be *D*. However, we will show that $D \circ V$ may be compositional even if there is a sense in which *both D* and *V* are.

Consider, once more, the axioms of Łukasiewicz's 3-valued logic. It is clear that its acceptable truth valuations are not generally compositional. But each of them can be composed out of the compositional mapping with the range {1,½,0}, and a trivial mapping of this set on {**T**, **F**}. Is there a sense in which the latter mapping is non-compositional? Of course, there is. If we take {1,½,0} with the superstructure of the operations which are needed to interpret the logical operators of the logic, then there is no homomorphism of the resulting algebra into an algebra with the carrier {**T**, **F**} (this is, of course the reason why the logic is three-valued!). But from the viewpoint of merely the relationship between {1,½,0} and {**T**, **F**}, why on Earth should we require anything like that? The mapping taking 1 to **T** and the other two values to **F** is as perspicuous as a mapping could be.

In general, though, it is clear that if *h* is a homomorphism from an algebra A_1 to an algebra A_2 and *h*' is a homomorphism from A_2 to an algebra A_3, then $h' \circ h$ is bound to be a homomorphism from A_1 to A_3; it is equally clear that if *h* is a homomorphism from A_1 to A_2 and *h*' is a homomorphism from an algebra A_2' with the same carrier as A_2 to A_3, then $h' \circ h$ *need not* be a homomorphism from A_1 to A_3. Take the usual interpretation of S5, mapping the sentences of the modal propositional language on the subsets of a set *W* (of 'possible worlds'). This is obviously a homomorphism from the algebra <*S*, <∧, ∨, ¬, □>>, where *S* is the set of sentences of the language and ∧, ∨, ¬ and □ are the obvious syntactic operations generated by the corresponding logical connectives, to the algebra <Pow(*W*), <∩, ∪, ~, Cl>>, where ∩, ∪ and ~ are the set-theoretical operations of intersection, union and complement and Cl is the 'closure' operator defined by Cl(x) = W iff x = W and Cl(x) = ∅ otherwise. Now given a $w \in W$, we can define the mapping of Pow(W) on the set {**T**,**F**} by $h(x)$ = T iff $w \in x$. Also this mapping is a homomorphism 'from Pow(W) to {**T**,**F**}'[7]. However *it is not a homomorphism of* <Pow(W), <∩, ∪, ~, Cl>>, and as a consequence, the composition of these two mappings is *not* a homomorphism from <S, <∧, ∨, ¬, □>> to an algebra with the carrier {**T**,**F**}. This is clear from the fact □ is not generally interpretable in terms of a truth table (more precisely: there is no function f over {**T**,**F**} such that the truth value of □s would be generally the value of f for the truth value of s).

We claimed that any valuation of a logical language *must* be a homomorphism – for the potential infinity of the sentences of the language exists solely via the underlying vocabulary & grammar. But this is usually not true for the values constituting the range of the valuation. Take the most extreme case when the range consists of a very small number of objects – such as the 1, ½ and 0 of a three-valued logic. There is no need to take the set as generated by a smaller basis; i.e. if we are to see it as an algebra, then it comes naturally to see it as a trivial one, without any operations. True, when we consider mappings from a language to it, we need to supplement it by the operations which would serve as counterparts of the operations of the algebra of expressions; however, if we consider the mapping from it to the set {**T**,**F**} that maps 1 on T and the other two values on F, there is no need to assume any nontrivial structure.

This leads to the situation that when considering a mapping from a language to a range of values, and then, when considering a mapping from the range to {**T**,**F**} we naturally think of *different* algebras over the range of values in each respective case, as illustrated by the following picture:



[7] Of course a homomorphism is not from a set to a set, but rather from an algebra to an algebra. But in this context it is natural to think precisely about a homomorphism from Pow(W) to {**T**,**F**} in the sense of a homomorphism from *some* algebra over Pow(W) to an algebra over {**T**,**F**}.

The higher dashed elipse represents the algebra constituted over the expressions by means of the grammatical rules; the lower dashed elipse represents the algebra constituted by the rules' counterparts over the intermediary values. *V* maps the former on the latter. *D*, on the other hand, maps the intermediary values (without the superstructure) directly on {**T**,**F**}.

As a result, we have two mappings which are 'unproblematic' and their composition, which is 'problematic' (= not a homomorphism). In this sense, we can speak about solving the problem of compositionalization by means of *compositional decomposition*: for languages which do not have 'unproblematic' (= compositional) truth-valuations there may be pairs of 'unproblematic' functions, with an intermediary range of values, the composition of which yields the truth-valuations. Here we will avoid discussion of whether the decomposition of truth-valuation yields *always* 'unproblematic' functions – in the case where the range of intermediary values is infinite, this is not obvious. Instead we will concentrate on the problem of the nature of the intermediary values.

Truth-values vs. Meanings

In some cases, the intermediary values were explicitly introduced as the result of the consideration that not every sentence can be seen as either true or false. Most of the traditional three-valued logics were driven by the recognition of the fact that natural language is full of sentences which do not deserve to be considered either true, or false: sentences grammatically well-formed, but semantically flawed (like Carnap's *Caesar is a prime* or Chomsky's *Colorless green ideas sleep furiously*); sentences with presuppositions which need not be fulfilled (like Russell's *The king of France is bald*) sentences with indexicals (like *I am hungry*) sentences about the future (like Aristotle's *There will be a naval battle tomorrow*) etc. In a similar way, the point of departure of paraconsistent logic was the acceptance of the idea that a sentence can not only be neither true, nor false, but also both true and false; and the point of departure of fuzzy logic was the embracement of the idea that truth and falsity come in degrees. Hence in these cases seeing the surplus values as *truth*-values has some independent motivation.

However, many logics were first constituted as axiomatic systems and were in (shorter or longer) pursuit of semantics. This was the case of intuitionist logic, modal logics, relevant logics etc. The trouble with all of them was that they (of course) did not generally allow for compositional truth-valuations: thus, there was no general way of 'computing' the truth-value of $\Box A$ from that of *A*; and no general way of computing the truth-value of the relevant implication $A \to B$ from those of *A* and *B*.

This may seem to indicate that the values arising out of the 'compositional decomposition' of such deductive systems should by taken as a merely instrumental matter – not as further truth values, but merely our expedient for solving the separation problem. But there remains, I think, also another option, which comes to light if we return to the case of modal logic.

We saw that the first step of the compositionalization looked like a refinement of the truth value *truth* - its splitting up into *necessary truth* and *contingent truth*. Should we not still see the ensuing values as *truth values*? But then compositionalization went on: we had to further refine these values; so that the final refinement was very fine-grained, and representable as classes of 'possible worlds'. In this shape, the new values seem to be explicative of something like the *truth conditions*; hence they start to resemble *meanings* much more than truth values. Hence, could it be that the values brought to life by compositionalization are, at least in some cases, (explications of) meanings?

To be able to answer this question, we would need to be clear about what meanings are; and this, given the vast and divergent literature about the topic seems to be next to impossible. But we may start from formal languages, the semantics of which is a much more perspicuous matter. In particular, when we generalize over the usual languages of formal logic, we can say[8] that a semantic interpretation can be generally conceived of as mapping of expressions on some non-linguistic (usually set-theoretical) objects – which we will call *denotations* – such that:

(i) the mapping is a homomorphism from the algebra of expressions to an algebra over the denotations[9];

(ii) the carrier of the algebra of denotations contains some 'distinguished' elements (which are needed to define the concept of a *satisfying* interpretation, or a *model*, of a given sentence or theory);

(iii) the algebra of denotations is in some respects 'considerably simpler' than that of the expressions (we need to block such mappings as the identity one from counting as a semantic interpretation – but in fact this need be no more complicated than necessary to do justice to (i) and (ii)).

Now it is easy to see that these general conditions accord with the constitutive conditions of an intermediary-values-assignment as discussed above. Namely:

(1) We required that the assignment is to be compositional, i.e. to be a homomorphism.

(2) We required that they be able to yield us truth-valuations of sentences, via a mapping of the intermediary values on the two truth-values; hence we *distinguished*, in effect, those of the values that are mapped on the *truth*.

(3) We required that the set of intermediary values should be 'the smallest possible', ie. that no its proper subset would do.

I think that this remarkable agreement establishes the claim that what we are in fact doing when introducing the intermediary values and the mapping of expressions on them is

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[8] As I have argued at length elsewhere (see Peregrin, 1994).

[9] To my knowledge, this algebraic picture was first presented by Montague (1970).

introducing a formal semantics – explicating the 'meanings' of the expressions of the language we are dealing with.

To avoid misunderstanding, not all formal languages possess anything of the nature of the meanings of expressions of natural language. The semantics of classical logic is restricted to *extensions*, which are notoriously bad explications of meanings[10]. However, many non-classical logics, in their effort to capture various 'non-extensional' inferential patterns known from natural language, require semantic interpretation which is much closer to that of natural language. (And it is precisely the point of this paper that the farther a logic extends beyond the boundaries of the 'extensional core' of natural language, the closer its semantic interpretation is to the intuitive concept of meaning.)

Can we say that (i)-(iii) are not only constitutive of the concept of semantic interpretation for formal languages, but rather of the very concept of meaning? I think that insofar as this question can be answered at all (due to the notorious vagueness of the term "meaning"), it should be answered positively. In particular, I think that the following three intuitions are sound:

(a) Meaning is essentially compositional. (Some semanticists object to this, but as I have argued elsewhere,[11] I do not think that the concept of meaning is extricable from compositionality.)

(b) Meaning co-determines truth. The truth value of a sentence is in general the result of two factors: what does the sentence mean and how are things in the world. (In contrast to the non-empirical languages of mathematics, meaning-assignment does not generally determine truth *alone* , also the state of the world matters – but this does not compromise the point made.)

(c) As many philosophers have – and I think with right – argued, meaning can (or should be generally construed) as a kind of 'contributions to truth'. (The list of the supporters of this view starts at least from Frege (if we do not want to go as far back as to Leibniz) and culminates with Donald Davidson)[12]. It follows that meanings are *nothing more* that what is needed to compositionally yield truth[13]. (Again, in the case of natural language, in contrast to formal ones, we will have also *empirical* truth and hence there will be also 'a contribution from the world'.)

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[10] As documented by Carnap (1947) and a host of his followers.

[11] See Peregrin (2005).

[12] See especially Davidson (1984).

[13] I presented the argument that the arousal of meaning out of truth can be reconstructed in terms of 'compositionalization' in Peregrin (1997); see also Peregrin (2001, Chapter 4.)

Conclusion

Although it is clear that sticking to merely two truth values does not seem to suffice to tackle the separation problem, the question whether there are, therefore, more than two *truth* values is not easy to answer. In particular, I think that though there are cases of many-valued interpretations of logical calculi which are directly motivated by the rejection of the *excluded middle*, in many cases the surplus values arise as more or less instrumental entities and are finally much more akin to *meanings* than to some new truth values.

Moreover, I think that this emergence of semantics out of the compositionalization of the true/false distinction reveals an interesting (though by no means unknown) aspect of the nature of semantics – the entanglement of meaning and truth. Within the context of many-valued logics this entanglement acquires almost the shape of the Hegelian change of quantity into quality – truth values, by refinement, become something else, *viz.* meanings.

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